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Effect of the medium viscosity on sound propagation and attenuation in ducts

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Abstract

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The viscosity of the medium plays an important role in defining the characteristics of sound wave propagation in ducts. This effect, due to the difficulty of analysis, has been either completely neglected in the literature or considered only approximately. One common assumption has been that viscosity would affect the inviscid acoustic wave indirectly through its effect on the mean flow.

In this study, a mathematical model is constructed to describe the physical problem in its general form without imposing assumptions a priori. The set of equations describing the model are solved in a two-dimensional duct. The losses due to thermal conductivity of the medium are neglected in order to focus on that due to viscosity. The physical quantities have been conveniently expressed in only two nondimensional quantities, a frequency or wave number bK and a Reynolds number Re. The effect of these two parameters on the propagation and attenuation constants was studied in detail for a wide range of bK and Re. Results were obtained for the zeroth- and higher-order modes. The huge amount of numerical results obtained led to rather interesting conclusions.

Keywords: Duct acoustics; viscosity effect; propagation and attenuation of sound waves.

1. Introduction

Viscosity plays an important role in the field of duct acoustics. The field of duct acoustics concerns itself with the study of the propagation and attenuation characteristics of sound waves in ducts in all cases of the presence or absence of a mean flow.

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The viscosity of the medium inside the duct imposes a no-slip condition at the duct boundaries. Moreover, viscosity of the medium together with its thermal conductivity are responsible for the natural attenuation of the sound waves.

The effect of viscosity on the attenuation of acoustic modes in ducts was first discussed by Rayleigh in 1945 [13].

During the early 1950s, the problem of the effect of viscosity and heat conductivity on the acoustic wave propagation in circular, rectangular and two-dimensional ducts that carry no mean flow received considerable attention.

Extension of Rayleigh's work, alternative formulation of the problem and experimental results appeared in the literature of that period ([1–3,15] are good examples).

Later, during the 1960s and early 1970s, the interest focused on the solution of the duct acoustic problem in the presence of mean flow in the duct. The interest in this area was promoted by the need to reduce the aircrafts' engine noise that became a major public and governmental concern.

The author [4–8], among others, studied sound propagation in lined ducts (i.e., ducts with finite admittance at the walls) in the presence of viscous mean flow.

The excellent review [11] refers to over 150 references and covers all aspects of duct acoustics.

In the bulk of the literature so far, the effect of viscosity on the acoustics in ducts that carry mean flow was limited to its effect on the velocity profile of the mean flow. The acoustic disturbance was considered inviscid with the unrealistic slip along the duct boundaries. One exception to this is [10] in which the acoustic quantities are expressed as the sum of an inviscid part and a boundary layer part insignificantly outside a thin layer next to the wall. The problem is reduced to solving the inviscid acoustic case but with a modified specific wall admittance.

The approach taken in the current research is the formulation of the problem in its general form to include all variables without imposing assumptions a priori. The goal is to provide depth to the understanding of the transmission phenomena and to examine the role played by the different parameters and their effects on the characteristics of the acoustic wave.

In the present study, a two-dimensional duct is considered. The duct consists of two parallel, infinite, rigid planes at a distance $2b$ as shown in Fig. 1.

The choice of this particular geometry provides simplicity in setting the boundary conditions; meanwhile, all the trends of the effect of the different parameters would be representative of any other duct geometry as experience in this field indicates.

The physical quantities have been expressed conveniently in only two nondimensional quantities, a frequency number bK and a Reynolds number Re. The effect of these two parameters on the propagation number and the attenuation number have been investigated for zeroth- and higher-order symmetric and anti-symmetric modes.

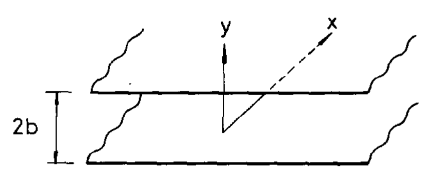

Fig. 1. Duct geometry.

2. Analytical model

2.1. Basic equations

The flow field inside the duct is governed by five basic equations [14].

(1) *Continuity equation*:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0, \tag{1}$$

where ρ is the density, t is the time, u and v are the velocities in the x- and y-directions respectively.

(2) *Navier–Stokes equation*: x-component of the momentum:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}-\tfrac{2}{3}\mu\,\nabla\cdot\boldsymbol{\omega}\right)+\frac{\partial}{\partial y}(\mu\,\nabla\cdot\boldsymbol{\omega}); \tag{2}$$

y-component of the momentum:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y}-\tfrac{2}{3}\mu\,\nabla\cdot\boldsymbol{\omega}\right)+\frac{\partial}{\partial x}\mu\,\nabla\cdot\boldsymbol{\omega}), \tag{3}$$

where p is the pressure, μ is the viscosity, $\boldsymbol{\omega}$ is the velocity vector and $\nabla\cdot\boldsymbol{\omega}\equiv\partial u/\partial x+\partial v/\partial y$.

(3) *Energy equation*: the energy equation for a perfect gas can be written as

$$\rho C_{\mathrm{p}}\frac{\mathrm{d}T}{\mathrm{d}t}=\frac{\partial p}{\partial t}+\left\{\frac{\partial}{\partial x}\left(k'\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k'\frac{\partial T}{\partial y}\right)\right\}+\mu\phi_{00},$$

where k' is the thermal conductivity of the gas, C_{p} is the specific heat at constant pressure, T is the temperature and ϕ_{00} is the dissipation function

$$2\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right\}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2.$$

In order to focus on the effects of viscosity on the attenuation of the acoustic waves, we choose to neglect the effect of thermal conductivity of the gas, and in what follows we set

$$k'=0.$$

Then, the energy equation for a perfect nonheat conducting gas would be

$$\rho C_{\mathrm{p}}\frac{\mathrm{d}T}{\mathrm{d}t}=\frac{\mathrm{d}p}{\mathrm{d}t}+\mu\phi_{00}. \tag{4}$$

(4) *State equation*:

$$P=\rho RT, \tag{5}$$

where R is the gas constant.

The steady state (mean) quantities are ρ_0, U, V, P_0, T_0 and ϕ_{00}.

2.2. Perturbation of the flow

Assume that an acoustic perturbation is superimposed over the flow, e.g., from a fan, a microphone or any other acoustic source; hence the new variables will be

$$\rho = \rho_0 + \bar{\rho}, \tag{6}$$

$$u = U + \bar{u}, \tag{7}$$

$$v = V + \bar{v}, \tag{8}$$

$$P = P_0 + \bar{P}, \tag{9}$$

$$T = T_0 + \bar{T}, \tag{10}$$

$$\phi_{00} = \Phi_{00} + \bar{\phi}_{00},$$

where the overbar denotes acoustic perturbation.

The equations (1)–(5) governing the flow including the acoustic wave components, after regrouping the terms for convenience, can be written as follows.

Equation (1) becomes

$$\left[\frac{\partial \rho_0}{\partial t} + \frac{\partial}{\partial x}(\rho_0 U) + \frac{\partial}{\partial y}(\rho_0 V)\right] + \left[\frac{\partial}{\partial t}\bar{\rho} + \frac{\partial}{\partial x}\rho_0 \bar{u} + \frac{\partial}{\partial x}(\bar{\rho} U) + \frac{\partial}{\partial y}(\rho_0 \bar{v}) + \frac{\partial}{\partial y}(\bar{\rho} V)\right]$$
$$+ \left[\frac{\partial}{\partial x}(\bar{\rho}\bar{u}) + \frac{\partial}{\partial y}(\bar{\rho}\bar{v})\right] = 0. \tag{11}$$

Equation (2) becomes

$$\left[\rho_0\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right) + \frac{\partial P_0}{\partial x} - \frac{\partial}{\partial x}\left(2\mu\frac{\partial U}{\partial x} - \tfrac{2}{3}\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right)\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y} + \mu\frac{\partial V}{\partial x}\right)\right]$$
$$+ \left[\rho_0\left(\frac{\partial \bar{u}}{\partial t} + U\frac{\partial \bar{u}}{\partial x} + V\frac{\partial u}{\partial y} + \bar{u}\frac{\partial U}{\partial x} + \bar{v}\frac{\partial U}{\partial y}\right) + \bar{\rho}\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right)\right.$$
$$\left. + \frac{\partial \bar{P}}{\partial x} - \frac{\partial}{\partial x}\left(2\mu\frac{\partial \bar{u}}{\partial x} - \tfrac{2}{3}\mu\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)\right) - \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} + \mu\frac{\partial \bar{v}}{\partial x}\right)\right]$$
$$+ \left[\rho_0\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) + \bar{\rho}\left(\frac{\partial \bar{u}}{\partial t} + U\frac{\partial \bar{u}}{\partial x} + V\frac{\partial \bar{u}}{\partial y}\right) + \bar{u}\frac{\partial U}{\partial x} + \bar{v}\frac{\partial U}{\partial y}\right.$$
$$\left. + \bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right)\right] = 0. \tag{12}$$

Equation (3) becomes

$$\left[\rho_0\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right) + \frac{\partial P_0}{\partial y} + \frac{\partial}{\partial y}\left(2\mu\frac{\partial V}{\partial y} - \tfrac{2}{3}\mu\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\right) - \frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial y} + \mu\frac{\partial V}{\partial x}\right)\right]$$

$$+\left[\rho_0\left(\frac{\partial\bar{v}}{\partial t}+U\frac{\partial\bar{v}}{\partial x}+V\frac{\partial\bar{v}}{\partial x}+\bar{u}\frac{\partial V}{\partial x}+\bar{v}\frac{\partial V}{\partial y}\right)+\bar{\rho}\left(\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}\right)+\frac{\partial\bar{P}}{\partial y}\right.$$
$$\left.-\frac{\partial}{\partial y}\left(2\mu\frac{\partial\bar{v}}{\partial y}-\tfrac{2}{3}\mu\left(\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}\right)\right)-\frac{\partial}{\partial x}\left(\mu\frac{\partial\bar{u}}{\partial y}+\mu\frac{\partial\bar{v}}{\partial x}\right)\right]$$
$$+\left[\rho_0\left(\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}\right)+\bar{\rho}\left(\frac{\partial\bar{v}}{\partial t}+\bar{u}\frac{\partial V}{\partial x}+\bar{v}\frac{\partial V}{\partial y}+\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+V\frac{\partial\bar{v}}{\partial y}+U\frac{\partial\bar{v}}{\partial x}\right)\right]=0. \tag{13}$$

Equation (4) becomes

$$\left[\rho_0C_\text{p}\frac{\mathrm{d}T_0}{\mathrm{d}t}-\frac{\mathrm{d}P_0}{\mathrm{d}t}-\mu\Phi_{00}\right]+\left[\rho_0C_\text{p}\frac{\mathrm{d}\bar{T}}{\mathrm{d}t}+\bar{\rho}C_\text{p}\frac{\mathrm{d}T_0}{\mathrm{d}t}-\frac{\mathrm{d}\bar{P}}{\mathrm{d}t}-\mu\bar{\phi}_{00}\right]=0. \tag{14}$$

Equation (5) becomes

$$[P_0-\rho_0RT_0]+\left[\bar{P}-\rho_0R\bar{T}-\bar{\rho}RT_0\right]-\left[(\bar{\rho}R\bar{T})\right]=0. \tag{15}$$

Simplify equations (11)–(15) taking into account that

(i) the equilibrium quantities satisfy the basic equations, hence the expression in the first square brackets in each equation vanishes;

(ii) we only consider a first-order perturbation, hence the product of perturbed quantities will be neglected, i.e., the expression in third square brackets in each equation will, also, be neglected;

(iii) $\frac{\partial}{\partial t}(\rho_0, U, T_0)=0,\quad \frac{\partial}{\partial y}\rho_0=\frac{\partial}{\partial x}\rho_0=0;$

(iv) the mean flow is parallel, i.e., $V=0$;

(v) $U=U(Y)$;

(vi) $\mu=$ constant.

Further, let us consider harmonic variation in time so that the basic acoustic quantities can be written as

$$\bar{\rho}=\rho(y)\exp(\mathrm{i}\omega t-\gamma x), \tag{16}$$
$$\bar{u}=u(y)\exp(\mathrm{i}\omega t-\gamma x), \tag{17}$$
$$\bar{v}=v(y)\exp(\mathrm{i}\omega t-\gamma x), \tag{18}$$
$$\bar{P}=P(y)\exp(\mathrm{i}\omega t-\gamma x), \tag{19}$$
$$\bar{T}=T(y)\exp(\mathrm{i}\omega t-\gamma x), \tag{20}$$

where ω is the acoustic frequency, γ is the complex wave number $\alpha'+\mathrm{i}\beta'$, α is the attenuation coefficient, β' is the phase difference coefficient, i is the imaginary number $\sqrt{-1}$. Substituting (16)–(20) in (11)–(15) and dropping the argument y, we get

$$\Omega\rho=\rho_0\left(\gamma u-\frac{\mathrm{d}v}{\mathrm{d}y}\right), \tag{21}$$

$$\mu \frac{\mathrm{d}^2 u}{\mathrm{d} y^2} + \tfrac{4}{3}\mu\gamma^2 u - \rho_0 \Omega u = \left(\rho_0 \frac{\mathrm{d}u}{\mathrm{d}y}\right) v + \tfrac{1}{3}\mu\gamma \frac{\mathrm{d}v}{\mathrm{d}y} - \gamma P, \tag{22}$$

$$\tfrac{4}{3}\mu \frac{\mathrm{d}^2 v}{\mathrm{d} y^2} + \mu\gamma^2 v - \rho_0 \Omega v = \tfrac{1}{3}\gamma\mu \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}P}{\mathrm{d}y}, \tag{23}$$

$$\Omega T = \frac{\Omega y}{\rho_0 C_\mathrm{p}} P + \frac{2\mu}{\rho_0 C_\mathrm{p}}\left(\frac{\mathrm{d}u}{\mathrm{d}y} - \gamma v\right)\frac{\mathrm{d}U}{\mathrm{d}y} \tag{24}$$

and

$$P = R(\rho_0 T + T_0 \rho), \tag{25}$$

where $\Omega = \mathrm{i}\omega - \gamma U$.

2.3. Normalization

It is much more practical to obtain solutions in nondimensional form. For this reason, our variables are expressed in terms of convenient basic geometric and ambient constant quantities.

2.3.1. Basic normalization quantities

Speed: ambient speed of sound $C_\mathrm{a} = (\bar{\gamma} R T_\mathrm{a})^{1/2}$, where $\bar{\gamma}$ is the ratio of specific heats $C_\mathrm{p}/C_\mathrm{v}$, C_v is the specific heat at constant volume and T_a is the static ambient temperature.

Length: b is the half width of the duct.

Time: b/C_a.

Density: ρ_a is the ambient density.

Pressure: $\rho_\mathrm{a} C_\mathrm{a}^2$.

Temperature: T_a is the static ambient temperature.

2.3.2. Normalized parameters

Normalized velocities: x-component $u_n = u(y)/C_\mathrm{a}$; y-component $v_n = v(y)/C_\mathrm{a}$; mean flow $M = U(y)/C_\mathrm{a}$ is the Mach number.

Normalized length: $y_n = y/b$.

Normalized density: $\rho_n = \rho(y)/\rho_\mathrm{a}$, $\rho_{0n} = \rho_0/\rho_\mathrm{a}$.

Reynolds number: $\mathrm{Re} = \rho_\mathrm{a} C_\mathrm{a} b/\mu$.

Normalized pressure: $P_n = P(y)/\rho_\mathrm{a} C_\mathrm{a}^2$.

Normalized temperature: $T_n = T(y)/T_\mathrm{a}$, $T_{0n} = T_0/T_\mathrm{a}$ and $\Omega(y) = \mathrm{i}\omega - \gamma U = \mathrm{i}\omega(1 - kM)$ where $k = \gamma/\mathrm{i}K$ and $K = \omega/C_\mathrm{a}$. Let $\Omega_n = \Omega(y)b/C_\mathrm{a}$; then $\Omega_n = \mathrm{i}bK(1 - kM)$.

Note that the presence of the imaginary number i will cause the basic parameters u, v, $P, \dots,$ etc. to be complex.

2.3.3. Normalized governing equations

Substituting the normalized parameters in the basic governing equations (21)–(25), we get the five governing equations — after dropping the subscript n — in the form

$$\Omega \rho = \rho_0 \left(\delta u - \frac{\mathrm{d}v}{\mathrm{d}y}\right), \tag{26}$$

$$\frac{1}{\mathrm{Re}}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+\left(\frac{4}{3}\frac{\delta^2}{\mathrm{Re}}-\rho_0\Omega\right)u=\rho_0 v\frac{\mathrm{d}M}{\mathrm{d}y}+\frac{\delta}{3\,\mathrm{Re}}\frac{\mathrm{d}v}{\mathrm{d}y}-\delta P, \tag{27}$$

$$\frac{4}{3\,\mathrm{Re}}\frac{\mathrm{d}^2v}{\mathrm{d}y^2}+\left(\frac{\delta^2}{\mathrm{Re}}-\rho_0\Omega\right)v=\frac{\delta}{3\,\mathrm{Re}}\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\mathrm{d}P}{\mathrm{d}y}, \tag{28}$$

$$\rho_0\Omega T\frac{1}{(\bar{\gamma}-1)}=\Omega P+\frac{2}{\mathrm{Re}}\left(\frac{\mathrm{d}u}{\mathrm{d}y}-\delta v\right)\frac{\mathrm{d}M}{\mathrm{d}y}, \tag{29}$$

$$P=\frac{1}{\bar{\gamma}}(\rho_0 T+T_0\rho), \tag{30}$$

where

$$\delta=\gamma b=\mathrm{i}(bK)k=\delta_1+\mathrm{i}\delta_2. \tag{31}$$

These are the set of five simultaneous differential equations in the five normalized acoustic parameters P, u, v, T and ρ.

The problem can be further reduced by substituting (26) and (29) into (30) to eliminate ρ and T, and we end up with only three equations in the three unknowns P, u and v, as follows:

$$\frac{1}{\mathrm{Re}}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}+\left(\frac{4}{3}\frac{\delta^2}{\mathrm{Re}}-\rho_0\Omega\right)u=\rho_0 v\frac{\mathrm{d}M}{\mathrm{d}y}+\frac{\delta}{3\,\mathrm{Re}}\frac{\mathrm{d}v}{\mathrm{d}y}-\delta P, \tag{32}$$

$$\frac{4}{3\,\mathrm{Re}}\frac{\mathrm{d}^2v}{\mathrm{d}y^2}+\left(\frac{\delta^2}{\mathrm{Re}}-\rho_0\Omega\right)v=\frac{\delta}{3\,\mathrm{Re}}\frac{\mathrm{d}u}{\mathrm{d}y}+\frac{\mathrm{d}P}{\mathrm{d}y}, \tag{33}$$

$$\Omega p=\left\{\frac{2(\bar{\gamma}-1)}{\mathrm{Re}}\frac{\mathrm{d}M}{\mathrm{d}y}\right\}\frac{\mathrm{d}u}{\mathrm{d}y}+\left(\delta u-\frac{\mathrm{d}v}{\mathrm{d}y}\right)\rho_0 T_0-\left\{2\frac{(\bar{\gamma}-1)}{\mathrm{Re}}\frac{\mathrm{d}M}{\mathrm{d}y}\right\}\delta v. \tag{34}$$

2.3.4. Important special cases

The case of inviscid acoustic waves can be obtained by assuming $\mu=0$ and hence $\mathrm{Re}\to\infty$. The equation for pressure (34), using (32) and (33), can be written as

$$\frac{\mathrm{d}^2P}{\mathrm{d}y^2}+\frac{2k}{1-kM}\frac{\mathrm{d}M}{\mathrm{d}y}\frac{\mathrm{d}P}{\mathrm{d}y}+(bK)^2\left[(1-kM)^2-k^2\right]P=0,$$

which is the same equation obtained in [12] and which was subsequently used in most studies that assumed inviscid acoustic disturbance, including [4–8].

Further, if we assume no mean flow, then

$$M=\frac{\mathrm{d}M'}{\mathrm{d}y}=0,$$

and the above pressure equation is reduced to

$$\frac{\mathrm{d}^2p}{\mathrm{d}y^2}+(bK)^2\left[1-k^2\right]P=0,$$

which has the general solution

$$P = C\cos\left[bK\sqrt{1-k^2}\,y + \tau\right],$$

since $v = 0$ at $y = \pm 1$; hence from (33) we get

$$\frac{\mathrm{d}P}{\mathrm{d}y} = 0 \quad \text{at } y = \pm 1,$$

which results in

$$\gamma = \sqrt{\left\{-K^2 + N^2\left(\frac{\pi}{2b}\right)^2\right\}}.$$

This is the well-known solution, where N is the modal number that takes the values 0, 1, 2,... etc. for the zeroth, first and second, etc. modes respectively.

Note that no propagation occurs unless the frequency is higher than the cut-off frequency, i.e., γ is imaginary. The condition of propagation is then

$$bK > \tfrac{1}{2}N\pi.$$

3. Solution of the case of zero mean flow

In this case we have

$$M = M_0 = 0, \qquad \rho_0 = 1, \qquad T_0 = 1;$$

hence $\Omega = \mathrm{i}bK(l - kM) = \mathrm{i}bK$. Therefore, (32)–(34) become

$$\frac{1}{\mathrm{Re}}\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \left(\frac{4}{3}\frac{\delta^2}{\mathrm{Re}} - \mathrm{i}bK\right)u - \frac{\delta}{3\,\mathrm{Re}}\frac{\mathrm{d}v}{\mathrm{d}y} + \delta P = 0, \tag{35}$$

$$\frac{4}{3\,\mathrm{Re}}\frac{\mathrm{d}^2v}{\mathrm{d}y^2} + \left(\frac{\delta^2}{\mathrm{Re}} - \mathrm{i}bK\right)v - \frac{\delta}{3\,\mathrm{Re}}\frac{\mathrm{d}u}{\mathrm{d}y} - \frac{\mathrm{d}P}{\mathrm{d}y} = 0, \tag{36}$$

$$(\mathrm{i}bK)P - \delta u + \frac{\mathrm{d}v}{\mathrm{d}y} = 0. \tag{37}$$

Equation (37) gives P in terms of u, v as follows:

$$P = \frac{\delta u}{\mathrm{i}bK} - \frac{1}{\mathrm{i}bK}\frac{\mathrm{d}v}{\mathrm{d}y}. \tag{38}$$

This, when substituted back in equations (35) and (36), results in two simultaneous equations in u, v as follows:

$$\frac{1}{\mathrm{Re}}\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + \left(\frac{4}{3}\frac{\delta^2}{\mathrm{Re}} - \mathrm{i}bK + \frac{\delta^2}{\mathrm{i}bK}\right)u - \left(\frac{\delta}{3\,\mathrm{Re}} + \frac{\delta}{\mathrm{i}bK}\right)\frac{\mathrm{d}v}{\mathrm{d}y} = 0, \tag{39}$$

$$\left(\frac{4}{3\,\mathrm{Re}} + \frac{1}{\mathrm{i}bK}\right)\frac{\mathrm{d}^2v}{\mathrm{d}y^2} - \left(\frac{\delta}{3\,\mathrm{Re}} + \frac{\delta}{\mathrm{i}bK}\right)\frac{\mathrm{d}u}{\mathrm{d}y} + \left(\frac{\delta^2}{\mathrm{Re}} - \mathrm{i}bK\right)v = 0. \tag{40}$$

After a lengthy mathematical manipulation, a solution for u, v is found to be of the form

$$u(y) = A_1 e^{\alpha y} + A_2 e^{-\alpha y} + A_3 e^{\beta y} + A_4 e^{-\beta y} \tag{41}$$

and

$$v(y) = \lambda\{\psi A_1 e^{\alpha y} - \psi A_2 e^{-\alpha y} + \phi A_3 e^{\beta y} - \phi A_4 e^{-\beta y}\}, \tag{42}$$

where

$$\begin{aligned}
&\alpha^2 = \left(\tfrac{1}{4}C_2^2 - C_0\right)^{1/2} - \tfrac{1}{2}C_2, \qquad \beta^2 = -\left(\tfrac{1}{4}C_2^2 - C_0\right)^{1/2} - \tfrac{1}{2}C_2,\\
&C_2 = \frac{a_2}{a_4}, \qquad C_0 = \frac{a_0}{a_4},\\
&a_4 = \frac{4}{3\ \mathrm{Re}} + \frac{1}{\mathrm{i}\ \mathrm{Re}(bK)}, \qquad a_2 = \frac{8\delta^2}{3\ \mathrm{Re}^2} + \frac{2\delta^2}{\mathrm{i}\ \mathrm{Re}(bK)} - \mathrm{i}\frac{7(bK)}{3\ \mathrm{Re}} - 1,\\
&a_0 = \frac{4\delta^2}{3\ \mathrm{Re}^2} - \mathrm{i}\frac{7\delta^2(bK)}{3\ \mathrm{Re}} + \frac{\delta^4}{\mathrm{i}\ \mathrm{Re}(bK)} - (bK)^2 - \delta^2,\\
&\lambda = \left(\frac{\delta}{3\ \mathrm{Re}} + \frac{\delta}{\mathrm{i}bK}\right)^{-1},\\
&\psi = \frac{\alpha}{\mathrm{Re}} + \frac{1}{\alpha}\left(\frac{4\delta^2}{3\ \mathrm{Re}} - \mathrm{i}bK + \frac{\delta^2}{\mathrm{i}bK}\right), \qquad \phi = \frac{\beta}{\mathrm{Re}} + \frac{1}{\beta}\left(\frac{\delta^2}{3\ \mathrm{Re}} - \mathrm{i}b_K + \frac{\delta^2}{\mathrm{i}bK}\right).
\end{aligned} \tag{43}$$

To get the values of the constants A_1–A_4 and the value of $\delta = \delta_1 + \mathrm{i}\delta_2$, we make use of the boundary conditions

$$u(\pm 1) = 0, \qquad v(\pm 1) = 0. \tag{44}$$

Applying (41) and (42) at the boundaries, we get

$$\begin{aligned}
&A_1 e^{\alpha} + A_2 e^{-\alpha} + A_3 e^{\beta} + A_4 e^{-\beta} = 0,\\
&A_1 e^{-\alpha} + A_2 e^{\alpha} + A_3 e^{-\beta} + A_4 e^{\beta} = 0,\\
&A_1 e^{\alpha}\psi - A_2 e^{-\alpha}\psi + A_3 e^{\beta}\phi - A_4 e^{-\beta}\phi = 0,\\
&A_1 e^{-\alpha}\psi - A_2 e^{\alpha}\psi + A_3 e^{-\beta}\phi - A_4 e^{\beta}\phi = 0.
\end{aligned}$$

Hence, we have four homogeneous equations in four unknowns, namely A_1–A_4. Therefore, to have a nontrivial solution, the determinant Δ of the coefficients of these unknowns must vanish, where

$$\Delta = \begin{vmatrix} e^{\alpha} & e^{-\alpha} & e^{\beta} & e^{-\beta} \\ e^{-\alpha} & e^{\alpha} & e^{-\beta} & e^{\beta} \\ e^{\alpha}\psi & -e^{-\alpha}\psi & e^{\beta}\phi & -e^{-\beta}\phi \\ e^{-\alpha}\psi & -e^{\alpha}\psi & e^{-\beta}\phi & -e^{\beta}\phi \end{vmatrix} = 0. \tag{45}$$

Expanding and simplifying, the above condition can be written as

$$\Delta = 4\{\phi \sinh\alpha \cosh\beta - \psi \cosh\alpha \sinh\beta\}\{\psi \sinh\alpha \cosh\beta - \phi \cosh\alpha \sinh\beta\} = 0. \tag{46}$$

For given parameters bK and Re, there are discrete-eigenvalues-values for $\delta = \delta_1 + i\delta_2$ that satisfy (46). A Newton–Raphson iteration scheme was used to numerically find δ_1 and δ_2. Whenever the iteration converges to eigenvalues δ_1 and δ_2, they are used to find out the corresponding eigenfunctions u, v and p [9] to insure that the solution converges to the right mode.

The following section describes and discusses the results obtained for both δ_1 and δ_2 for the different propagation modes and for a wide range of the two basic parameters Re and bK.

4. Analysis of results

As a reminder, we should recall that according to our model the acoustic wave is expressed in the nondimensional form

$$S = S(y)\exp[i(Kbt - \delta_2 x) - \delta_1 x],$$

where $S(y)$ is a complex quantity that varies across the duct width. S represents any physical quantity associated with the sound wave, i.e., it could be sound pressure P, axial particle velocity u, transverse particle velocity V, temperature T or density ρ.

During the course of this study a huge amount of numerical data was generated. A range of bK from 10 to 25 and a range of Re from 500 to 10^7 were considered. For a wide range of combinations of these two physical parameters the eigenvalues δ_1 (the attenuation number) and δ_2 (the propagation number) were computed for the zeroth, first, second, third and fourth modes.

In the following, the effects of the medium viscosity, as represented by the Reynolds number Re, on the propagation and attenuation characteristics of the sound wave are discussed.

4.1. *The effect of viscosity on the propagation number δ_2*

From the analysis presented above, it is shown that for inviscid medium, i.e., as Re $\to \infty$, the propagation constant δ_2 can be computed from the formula

$$\delta_2^* = \sqrt{(bK)^2 - (\tfrac{1}{2}N\pi)^2}\,.$$

Table 1
The propagation number δ_2 for the zeroth mode

$bK = 15$ ($\delta_2^* = 15.0000$)			$bK = 20$ ($\delta_2^* = 20.0000$)			$bK = 25$ ($\delta_2^* = 25.0000$)		
Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*
750	15.0053	1.0004	550	19.9005	0.9950	980	24.9644	0.9986
850	15.0118	1.0008	1000	19.8963	0.9948	1250	24.9251	0.9970
1000	15.0173	1.0012	2000	20.0110	1.0005	1500	24.9241	0.9970
1250	15.0213	1.0014	3000	20.0172	1.0009	2000	24.9164	0.9967
1500	15.0227	1.0015	4000	20.0179	1.0009	4000	25.0102	1.0004
1650	15.0230	1.0015	50000	20.0175	1.0009	10^4	25.0141	1.0006

Table 2
The propagation number δ_2 for the first mode

$bK = 10$ ($\delta_2^* = 9.8759$)			$bK = 15$ ($\delta_2^* = 14.9175$)			$bK = 25$ ($\delta_2^* = 24.9506$)		
Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*
1650	9.9284	1.0053	1650	14.9777	1.0040	500	24.7015	0.9900
3000	9.9155	1.0040	3000	14.9647	1.0032	1000	24.9718	1.0008
10^4	9.8978	1.0022	10^4	14.9444	1.0018	4000	25.0102	1.0024
10^5	9.8828	1.0007	10^5	14.9261	1.0006	10^5	24.9617	1.0004
10^6	9.8781	1.0002	10^6	14.9202	1.0002	10^6	24.9541	1.0001
10^7	9.8766	1.0001	10^7	14.9184	1.0001	10^7	24.9517	1.0000

Table 3
The propagation number δ_2 for the second mode

$bK = 10$ ($\delta_2^* = 9.4937$)			$bK = 15$ ($\delta_2^* = 14.6673$)			$bK = 20$ ($\delta_2^* = 19.7517$)		
Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*
1150	9.5481	1.0057	1150	14.7400	1.0050	1150	19.8527	1.0051
2000	9.5415	1.0050	2000	14.7302	1.0043	2000	19.8343	1.0042
10^4	9.5150	1.0022	10^4	14.6943	1.0018	10^4	19.7837	1.0016
10^5	9.5004	1.0007	10^5	14.6758	1.0006	10^5	19.7616	1.0005
10^6	9.4958	1.0002	10^6	14.6700	1.0002	10^6	19.7548	1.0002
10^7	9.4944	1.0001	10^7	14.6682	1.0001	10^7	19.7527	1.0000

Table 4
The propagation number δ_2 for the third mode

$bK = 15$ ($\delta_2^* = 14.2406$)			$bK = 20$ ($\delta D_2^* = 19.4369$)			$bK = 25$ ($\delta_2^* = 24.5519$)		
Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*
750	14.3374	1.0068	750	19.5618	1.0068	750	24.7003	1.0060
1500	14.3085	1.0044	1500	19.5208	1.0043	1500	24.6543	1.0042
3000	14.2884	1.0034	3000	19.4946	1.0030	3000	24.6197	1.0028
10^4	14.2666	1.0018	10^4	19.4679	1.0016	10^4	24.5873	1.0014
10^6	14.2432	1.0002	10^6	19.4400	1.0002	10^6	24.5553	1.0001
10^7	14.2413	1.0001	10^7	19.4379	1.0001	10^7	24.5529	1.0000

Table 5
The propagation number δ_2 for the fourth mode

$bK = 15$ ($\delta_2^* = 13.6206$)			$bK = 20$ ($\delta_2^* = 18.9874$)			$bK = 25$ ($\delta_2^* = 24.1976$)		
Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*	Re	δ_2	δ_2/δ_2^*
750	13.7093	1.0065	750	19.0966	1.0058	750	24.3272	1.0054
1500	13.6844	1.0047	1500	19.0655	1.0041	1500	24.2891	1.0038
3000	13.6660	1.0033	3000	19.0426	1.0029	3000	24.2615	1.0026
10^4	13.6455	1.0018	10^4	19.0175	1.0016	10^4	24.2321	1.0014
10^6	13.6231	1.0002	10^6	18.9904	1.0002	10^6	24.2010	1.0001
10^7	13.6214	1.0001	10^7	18.9884	1.0001	10^7	24.1986	1.0000

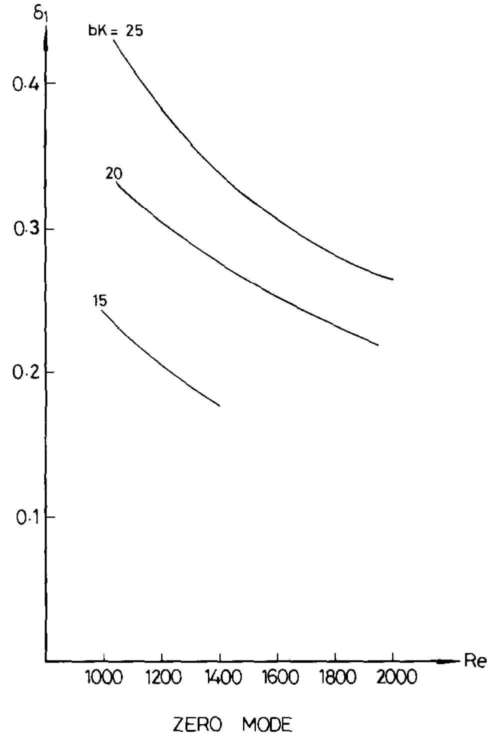

Fig. 2. The effect of Re on the attenuation of the zeroth mode.

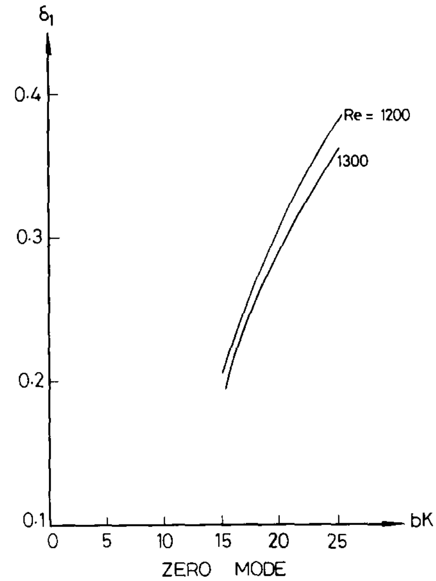

Fig. 3. The effect of bK on the attenuation of the zeroth mode.

For finite Re, δ_2 is computed numerically and shown in Tables 1–5 for the first five modes. These results show that the propagation number is a weak function of Re. At high Re (higher than 10^4) as Re increases, δ_2 approaches δ_2^* (its inviscid value) asymptotically from above.

For all practical purposes, the effect of viscosity on the wave propagation is negligible. This means that the speed of the wave propagation is, to a large extent, independent of the viscosity of the medium.

These findings extend and enhance the earlier conclusions of [6,8] that the wave propagation number is independent of not only viscosity of the medium but also the duct wall admittance and the profile of the viscous mean flow. The conclusion is that the propagation number for a given mode is dependent only of the nondimensional frequency bK and the average Mach number of the mean flow.

4.2. *The effect of viscosity on the attenuation number δ_1*

The effect of the medium viscosity on the attenuation of the zeroth mode is shown in Figs. 2 and 3.

Figure 2 shows that the attenuation of the sound wave due to the medium viscosity is substantial at relatively low Re. In order to clarify the significance of the numbers in these figures, we pay attention to the fact that a value of δ_1 of 0.35 would mean that the sound level

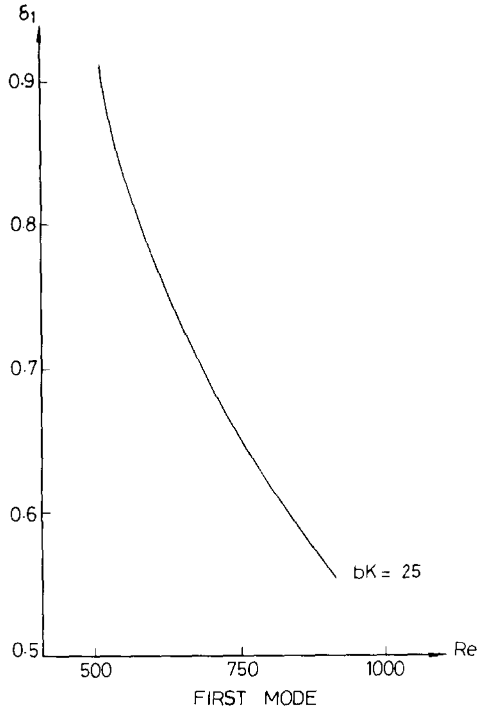

Fig. 4. The effect of Re on the attenuation of the first mode.

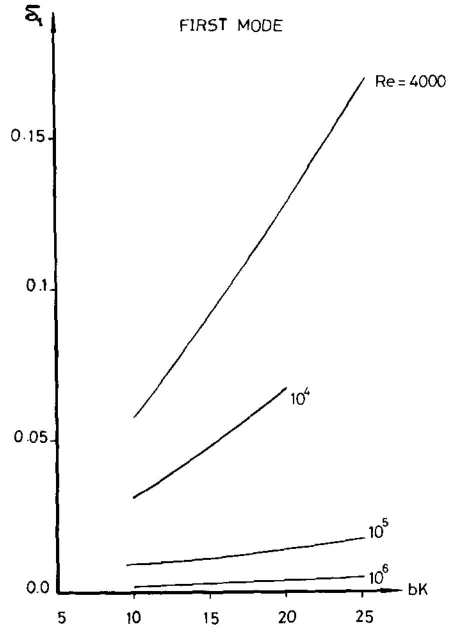

Fig. 5. The effect of bK on the attenuation of the first mode.

would drop to less than half its value in a distance equivalent to the duct width $2b$. Hence, neglecting the medium viscosity ρ in any situation of low Re would result in significant errors. Figure 2 also shows that attenuation decreases as Re increases. δ_1, and hence attenuation, is

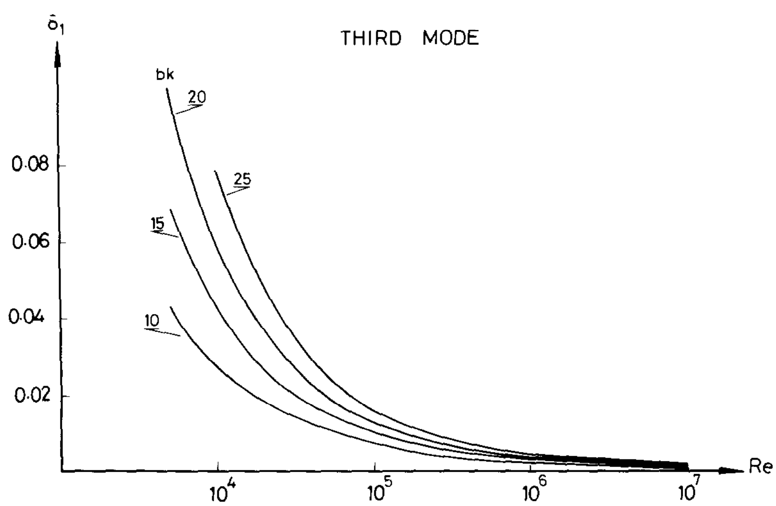

Fig. 6. The effect of Re on the attenuation of the third mode.

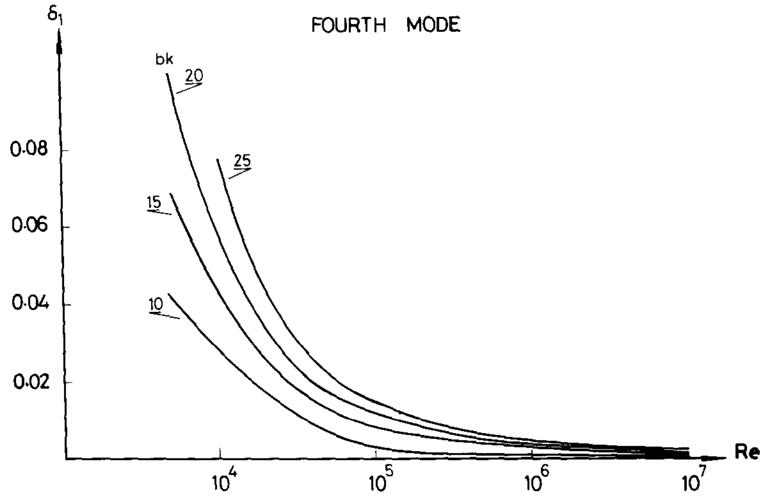

Fig. 7. The effect of Re on the attenuation of the fourth mode.

higher at high frequency bK. This fact is further illustrated in Fig. 3. At the same Re, attenuation increases rapidly as the sound frequency increases.

Figure 4 shows considerable attenuation of the first mode at Re = 500 and $bK = 25$, the level of the sound wave will drop to 15% of its value in a distance equivalent to one duct width. It, also, shows that attenuation decreases very rapidly as Re increases.

For a wider range of both Re and bK, Figs. 5 and 6 further illustrate the variation of δ_1. Figure 5 shows that as bK increases, the attenuation δ_1 increases. The rate of increase is much higher at lower Re. Figure 6 shows that for Re $> 10^6$ the attenuation tends to zero as Re increases. At these high Re the attenuation is practically independent of the frequency bK. For higher modes, the results of this study show that not only the trends shown in Figs. 6 and 7 are repeated, but also the rather very interesting fact that the attenuation values do not vary much at the same Re and bK.

Tables 6–9 show that

(a) at low Re and high bK, higher modes would suffer slightly more attenuation;

(b) at high Re, higher modes have slightly less attenuation at the same bK.

Table 6
The attenuation number δ_1, at $bK = 10$

Re	1st mode	2nd mode	3rd mode	4th mode
1650	0.1093	0.0979	0.0960	0.0960
3000	0.0703	0.0639	0.0621	0.0610
4000	0.0574	0.0524	0.0508	0.0494
5000	0.0492	0.0451	0.0436	0.0421
10^4	0.0310	0.0288	0.0276	0.0262
10^5	0.0079	0.0075	0.0070	0.0064
10^6	0.0023	0.0022	0.0021	0.0018
10^7	0.0007	0.0007	0.0006	0.0006

Table 7
The attenuation number δ_1, at $bK = 15$

Re	1st mode	2nd mode	3rd mode	4th mode
1650	0.1911	0.1638	0.1621	0.1630
3000	0.1173	0.1030	0.1016	0.1015
4000	0.0936	0.0831	0.0817	0.0814
5000	0.0788	0.0749	0.0693	0.0688
10^4	0.0473	0.0432	0.0423	0.0417
10^5	0.0106	0.0101	0.0098	0.0095
10^6	0.0029	0.0028	0.0028	0.0027
10^7	0.0009	0.0009	0.0008	0.0008

Table 8
The attenuation number δ_1, at $bK = 20$

Re	1st mode	2nd mode	3rd mode	4th mode
1500	0.3281	0.2622	0.2637	0.2663
3000	0.1471	0.1508	0.1492	0.1496
4000	0.1493	0.1201	0.1186	0.1186
5000	0.1167	0.1009	0.0995	0.0094
10^4	0.0673	0.0598	0.0589	0.0586
10^5	0.0135	0.0128	0.0126	0.0124
10^6	0.0035	0.0034	0.0033	0.0033
10^7	0.0013	0.0010	0.0010	0.0010

Table 9
The attenuation number δ_1, at $bK = 25$

Re	1st mode	2nd mode	3rd mode	4th mode
1500	–	–	0.3673	0.3744
3000	–	0.2046	0.2054	0.2067
4000	–	0.1631	0.1620	0.1625
5000	–	0.1364	0.1350	0.1352
10^4	–	0.0791	0.0780	0.0778
10^5	0.0167	0.0156	0.0154	0.0152
10^6	0.0041	0.0039	0.0039	0.0039
10^7	0.0017	0.0012	0.0012	0.0012

5. Conclusions

(1) In order to study the effect of medium viscosity on the characteristics of sound wave propagation and attenuation in ducts a mathematical model was constructed without introducing any assumptions a priori.

The set of equations describing our model is solved analytically as far as possible. The eigenvalues that represent the modal propagation and attenuation numbers were obtained numerically using FORTRAN 77 with real numbers represented in double-precision mode to insure maximal computational accuracy. Numerous numerical computational problems were encountered and were overcome.

(2) The physical quantities that govern the problem were conveniently grouped only in two nondimensional quantities, a Reynolds number Re and a frequency (or wave) number bK. The effect of these two parameters on the model propagation and attenuation characteristics is presented in both graphical and tabular form.

(3) The medium viscosity affects the propagation number only slightly. This is in agreement with earlier findings, see [6].

(4) The medium viscosity affects the wave attenuation significantly in particular at high frequency and low Reynolds number. Neglecting viscosity in such a situation would lead to significant errors.

(5) At the same Re and bK, the attenuation of different modes varies slightly. This leads us to expect that the profile of the sound wave changes slowly and only moderately along the duct length.

References

[1] R.E.J. Beatty, Boundary layer attenuation of higher-order modes in rectangular and circular tubes, *J. Acoust. Soc. Amer.* **22** (1950) 850–854.
[2] L. Cremer, Über die Akustische Grenzschicht vor starren Wanden, *Arch. Elektrischen Übertragung* **2** (1948) 136–139.
[3] R.F. Lambert, A study of the factors influencing the damping of an acoustical cavity resonator, *J. Acoust. Soc. Amer.* **25** (1953) 1068–1083.
[4] M.N. Mikhail, Noise shear flow interaction in lined ducts, M.Sc. Thesis, Carleton Univ., Ottawa, 1973.
[5] M.N. Mikhail and A.N. Abdelhamid, Shear flow effect on the propagation and attenuation of sound waves in an acoustically treated annular duct, Report No. ME 73-2, Faculty of Engrg., Carleton Univ., 1973.
[6] M.N. Mikhail and A.N. Abdelhamid, A rapid method for the solution of acoustic propagation in ducts containing shear flow, Report No. ME 73-3, Faculty of Engrg., Carleton Univ., 1973.
[7] M.N. Mikhail and A.N. Abdelhamid, Transmission and far-field radiation of sound waves in and from lined ducts containing shear flow, in: *AIAA Aero-Acoustics Conf.*, Seattle, WA, 1973; AIAA paper No. 73-1013.
[8] M.N. Mikhail and A.N. Abdelhamid, Transmission and far-field radiation of sound waves in and from lined ducts containing shear flow, in: H.T. Nagamatsu, Ed., *Aeroacoustics: Jet and Combustion Noise; Duct Acoustics,* Progress in Astronautics and Aeronautics **37** (MIT Press, Cambridge, MA, 1975) 353–371.
[9] M.N. Mikhail and M.R. El-Tantawy, The acoustic boundary layers: a detailed analysis, *J. Comput. Appl. Math.*, to appear.
[10] A.H. Nayfeh, Effects of the acoustic boundary layer on the wave propagation in ducts, *J. Acoust. Soc. Amer.* **54** (1973) 1737–1742.
[11] A.H. Nayfeh, J.E. Kaiser and D.P. Telionis, Acousts of aircraft engine-duct systems, *AIAA J.* **13** (1975) 130–153.
[12] D.C. Pridmore-Brown, Sound propagation in a fluid flowing through an attenuating duct, *J. Fluid Mech.* **4** (1958) 393–406.
[13] Lord Rayleigh, *Theory of Sound, Vol. II* (Dover, New York, 1945).
[14] H. Schlichting, *Boundary Layer Theory* (McGraw-Hill, New York, 1968).
[15] E.A.G. Shaw, The attenuation of the higher modes of acoustic waves in a rectangular tube, *Acustica* **3** (1953) 87–95.